Worksheet Implicit Diff.

① $x^3 + xy + y^3 = 1$

$3x^2 + x\frac{dy}{dx} + y + 3y^2\frac{dy}{dx} = 0$

$(x+3y^2)\frac{dy}{dx} = -3x^2 - y$

$\boxed{\frac{dy}{dx} = \frac{-3x^2-y}{x+3y^2}}$

② $y - (x)(\sin y) = 3$

$\frac{dy}{dx} - x\cos y\frac{dy}{dx} - \sin y = 0$

$(1 - x\cos y)\frac{dy}{dx} = \sin y$

$\boxed{\frac{dy}{dx} = \frac{\sin y}{1 - x\cos y}}$

③ $x + \tan(xy) = 0$

$1 + (\sec^2(xy))(x\frac{dy}{dx} + y) = 0$

$1 + x\sec^2(xy)\frac{dy}{dx} + y\sec^2(xy) = 0$

$x\sec^2(xy)\frac{dy}{dx} = -1 - y\sec^2(xy)$

$\frac{dy}{dx} = \boxed{\frac{-1 - y\sec^2(xy)}{x\sec^2(xy)}} = -\frac{\cos^2(xy)}{x} - \frac{y}{x}$

$= \boxed{\frac{-\cos^2(xy) - y}{x}}$

④ $y = xy + x^2 + 1$

When $x = -1$, $y = -y + 2$

$2y = 2$

$y = 1$

$\frac{dy}{dx} = x\frac{dy}{dx} + y + 2x$

$(1-x)\frac{dy}{dx} = y + 2x$

$\frac{dy}{dx} = \boxed{\frac{y+2x}{1-x}}\Bigg|_{(-1,1)} = \frac{1-2}{1--1} = \boxed{-\frac{1}{2}}$

⑤ $y^2 + 2y = 2x + 1$

$2y\frac{dy}{dx} + 2\frac{dy}{dx} = 2$

$(2y+2)\frac{dy}{dx} = 2$

$\boxed{\frac{dy}{dx} = \frac{1}{y+1} = (y+1)^{-1}}$

$\frac{d^2y}{dx^2} = -(y+1)^{-2}\frac{dy}{dx}$

$= -\frac{1}{(y+1)^2}\cdot\frac{1}{y+1}$

$= \boxed{-\frac{1}{(y+1)^3}}$

⑥ $x^3+y^3=8$

$3x^2+3y^2\frac{dy}{dx}=0$

$\boxed{\frac{dy}{dx}=-\frac{x^2}{y^2}}$ $=-\frac{x^2}{y^2}$

$$\frac{d^2y}{dx^2}=\frac{y^2(-2x)-(-x^2)(2y\frac{dy}{dx})}{y^4}$$

$$=\frac{-2xy^2+2x^2y\left(-\frac{x^2}{y^2}\right)}{y^4}\cdot\frac{y}{y}$$

$$=\boxed{\frac{-2xy^3-2x^4}{y^5}}=\frac{-2x(y^3+x^3)}{y^5}$$

$$=\frac{-2x(8)}{y^5}=\boxed{-\frac{16x}{y^5}}$$

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⑦ $y+\cos y=x+1 \quad 0\le y\le 2\pi$

(a) $\frac{dy}{dx}-\sin y\frac{dy}{dx}=1$

$(1-\sin y)\frac{dy}{dx}=1$

$\boxed{\frac{dy}{dx}=\frac{1}{1-\sin y},\ y\neq\frac{\pi}{2}}$

(b) Vertical tangent where $1-\sin y=0$, $1=\sin y$

when $y=\frac{\pi}{2}$

$\frac{\pi}{2}+0=x+1$

$\boxed{\frac{\pi}{2}-1=x}$

(c) $\frac{d^2y}{dx^2}=-(1-\sin y)^{-2}(-\cos y\frac{dy}{dx})$ using $\frac{dy}{dx}=(1-\sin y)^{-1}$

$$=\frac{\cos y}{(1-\sin y)^2}\cdot\frac{1}{1-\sin y}=\boxed{\frac{\cos y}{(1-\sin y)^3}}$$

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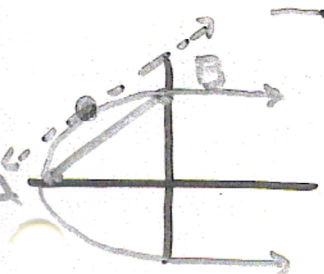
⑧ $y^2=4+x$ $\quad$ A $(-4,0)$, B $(0,2)$

$2y\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{2y}$



Slope of $\overline{AB}=\frac{2-0}{0--4}=\frac{1}{2}$

$\frac{1}{2y}=\frac{1}{2}$

$y=1$

$1=4+x$

$-3=x$ $\quad$ $\boxed{(-3,1)}$

⑨ $\frac{dy}{dx} = y^2(6-2x)$

$$\frac{d^2y}{dx^2} = y^2(-2) + (6-2x)\left(2y\,\frac{dy}{dx}\right)$$

$$= -2y^2 + 2y(6-2x)\cdot y^2(6-2x)$$

$$= -2y^2 + 2y^3(6-2x)^2$$

At $\left(3, \frac{1}{4}\right)$, $\frac{d^2y}{dx^2} = -2\left(\frac{1}{16}\right) + 2\left(\frac{1}{64}\right)(0) = \boxed{-\frac{1}{8}}$

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⑩ $xy^2 - x^3y = 6$

(a) $x\left(2y\frac{dy}{dx}\right) + y^2 - x^3\frac{dy}{dx} - 3x^2y = 0$

$$(2xy - x^3)\frac{dy}{dx} = 3x^2y - y^2$$

$$\boxed{\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}}$$

(b) If $x = 1$, $y^2 - y = 6$

$$y^2 - y - 6 = 0$$

$$(y-3)(y+2) = 0$$

$y = 3$ | $y = -2$

$(1, 3)$ | $(1, -2)$

At $(1, 3)$, $\frac{dy}{dx} = \frac{3(1)(3) - 9}{2(1)(3) - 1} = 0$ $\boxed{y = 3}$

At $(1, -2)$, $\frac{dy}{dx} = \frac{3(1)(-2) - 4}{2(1)(-2) - 1} = \frac{-10}{-5} = 2$ $\boxed{y + 2 = 2(x-1)}$

(c) $\frac{dy}{dx}$ is undefined when $2xy - x^3 = 0$ ← Tangent line is vertical

(We were given $xy^2 - x^3y = 6$)

$$x(2y - x^2) = 0$$

$x = 0$

$0 - 0 \neq 6$

$\boxed{\text{No such point on the curve}}$

$2y - x^2 = 0$ ←

$y = \frac{x^2}{2}$

$x\left(\frac{x^2}{2}\right)^2 - x^3\left(\frac{x^2}{2}\right) = 6$

$\frac{x^5}{4} - \frac{x^5}{2} = 6$ $\boxed{x = \sqrt[5]{-24}}$

$x^5 = -24$

(11) $\lim_{h \to 0} \dfrac{\tan\left(\frac{\pi}{4}+h\right)-1}{h} = f'\left(\frac{\pi}{4}\right)$ where $f(x) = \tan x$

$f'(x) = \sec^2 x$

$\boxed{f'\left(\frac{\pi}{4}\right) = 2}$

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(12) $\lim_{h \to 0} \dfrac{5(x+h)^2 - 5x^2}{h} = f'(x)$ where $f(x) = 5x^2$

$\boxed{f'(x) = 10x}$

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(13) $\lim_{h \to 0} \dfrac{(h+2)^6 - 64}{h} = \lim_{h \to 0} \dfrac{(2+h)^6 - 2^6}{h} = f'(2)$

where $f(x) = x^6$

$f'(x) = 6x^5$

$\boxed{f'(2) = 6(32) = 192}$

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(14) $\lim_{h \to 0} \dfrac{\tan(3(x+h)) - \tan(3x)}{h} = f'(x)$ where $f(x) = \tan(3x)$

$\boxed{f'(x) = 3\sec^2(3x)}$

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(15) $\lim_{x \to 2} \dfrac{2x^3 - 16}{x-2} = \lim_{x \to 2} \dfrac{2x^3 - 2(2)^3}{x-2} = f'(2)$

where $f(x) = 2x^3$

$f'(x) = 6x^2$

$\boxed{f'(2) = 6(4) = 24}$

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(16) $\lim_{x \to \frac{\pi}{2}} \dfrac{\sin x - 1}{x - \frac{\pi}{2}} = \lim_{x \to \frac{\pi}{2}} \dfrac{\sin x - \sin\frac{\pi}{2}}{x - \frac{\pi}{2}} = f'\left(\frac{\pi}{2}\right)$

where $f(x) = \sin x$

$f'(x) = \cos x$

$\boxed{f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0}$

(17) I. $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$

II. $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$

III. $\lim_{x \to a} \frac{f(x+h) - f(x)}{h} \neq f'(a)$ $\boxed{\text{(C) I and II only}}$

x	f(x)	g(x)	f'(x)	g'(x)
2	5	5	e	$\sqrt{2}$
5	2	8	π	7

(18) $h(x) = f(x) + g(x)$

$h'(x) = f'(x) + g'(x)$

$\boxed{h'(2) = e + \sqrt{2}}$

(19) $h(x) = f(x)\,g(x)$

$h'(x) = f(x)g'(x) + g(x)\,f'(x)$

$\boxed{h'(2) = 5\sqrt{2} + 5e}$

(20) $h(x) = \frac{f(x)}{g(x)}$

$h'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

$h'(2) = \frac{5e - 5\sqrt{2}}{5^2} = \boxed{\frac{e - \sqrt{2}}{5}}$

(21) $h(x) = f(g(x))$

$h'(x) = f'(g(x)) \cdot g'(x)$

$h'(2) = f'(5) \cdot g'(2)$

$= \boxed{\pi\sqrt{2}}$

(22) $h(x) = g(f(x))$

$h'(x) = g'(f(x)) \cdot f'(x)$

$h'(2) = g'(5) \cdot f'(2)$

$= \boxed{7e}$

(23) $h(x) = f(f(x))$

$h'(x) = f'(f(x)) \cdot f'(x)$

$h'(2) = f'(5) \cdot f'(2)$

$= \boxed{\pi e}$

(24) $h(x) = (f(x))^3$

$h'(x) = 3\,(f(x))^2 (f'(x))$

$h'(2) = 3\,(5)^2\,(e) = \boxed{75e}$